JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **29**, No. 2, May 2016
http://dx.doi.org/10.14403/jcms.2016.29.2.361

A REMARK ON FANO MANIFOLDS OF PICARD NUMBER ONE AND INDEX GREATER THAN ONE

Nam-Hoon Lee*

Abstract. This note is a remark on simple observation that the degree bound of Fano $(n+1)$-folds of Picard number one and index greater than one can be related with that of Fano n-folds of Picard number one and index one.

1. The observation

A *Fano n-fold* is a smooth projective variety X whose anticanonical divisor $-K_X$ is ample. The index of X is the largest integer m such that $-K_X = mH$ for some divisor H. The number $(-K_X)^n$ is called the degree and the divisor H (respectively the linear system $|H|$) is called the fundamental divisor (respectively the fundamental linear system) of the Fano n-fold X. Note that the indexes and degrees are invariant under deformation. If X has Picard number one, it is said to be of *the first species*. Iskovskikh classified Fano 3-folds of the first species in [1], where he assumed:

"The fundamental linear system $|H|$ contains a smooth surface"

This assumption was proved later by Shokurov ([2, 3]).

According to the Iskovskikh–Mori–Mukai classification of Fano 3-folds, there are 17 deformation families of Fano 3-folds of the first species and ten of them have index one. Among those ten families, five families contain Fano 3-folds that are cyclic covers over another Fano 3-fold. For example, consider the family of Fano 3-folds of the first species with index one and degree 10. Generically they are smooth sections of the Grassmannian $\mathrm{Gr}(2, 5)$ embedded by Plücker by a subspace of codimension two and quadric. But some members of the family are double covers over smooth sections of the Grassmannian $\mathrm{Gr}(2, 5)$ embedded by Plücker

Received February 01, 2016; Revised April 27, 2016; Accepted May 09, 2016.
2010 Mathematics Subject Classification: Primary 14J45.
Key words and phrases: Fano varieties, Calabi-Yau manifolds.
*This work was supported by 2013 Hongik University Research Fund.

by a subspace of codimension three, branched along surfaces of degree two. The base varieties are Fano 3-folds of the first species with index two and degree 40. Let us say that a family of Fano n-folds of first species is of *cyclic type* if it contain Fano n-folds that are cyclic covers over another Fano n-fold, branched along smooth divisors.

In this note, we relate the bound of degrees of Fano $(n+1)$-fold of the first species with index greater than one with that of degrees of Fano n-fold of the first species with index one under an assumption:

"Families of Fano n-folds of the first species with index greater than one contain Fano n-folds such that the linear system $|(r-1)H|$ contains a smooth divisor".

This assumption is somewhat weaker than the original assumption of Iskovskih and true for dimension two, three ([2, 3]) and four ([4, 5]). Actually the fundamental divisors of Fano n-folds of index greater than one are very ample at least for dimension $n \leq 4$. Moreover Fano n-fold of index $n+1$, n, $n-1$, $n-2$ are all classified ([7], [8], [5], [9]) and one can easily check that the assumption is true for those Fano n-folds.

Let

$$b(n,r) = \max\{\text{ degrees of Fano } n\text{-folds of the first species with index } r\},$$
$$b(n) = \max\{b(n,r) | r \in \mathbb{N}\} \quad \text{and}$$
$$\tilde{b}(n) = \max\{\text{ degrees of Fano } n\text{-folds of cyclic type with index one}\}.$$

It is known that $b(n) \leq \left(\frac{n^2+4n+3}{4}\right)^n$ ([6]).

If the fundamental divisor H is very ample (which is true for most of Fano 3-folds), the dimension of the fundamental linear system $|H|$ is the dimension of the projective space into which X is embedded. The number is invariant under deformation. Hence let

$$c(n,r) = \max\{\text{ dimensions of the fundamental linear systems of Fano } n\text{-folds of the first species with index } r\}.$$

THEOREM 1.1. *Under the assumption,*

(1) $b(n+1,r) \leq \frac{r^{n+1}}{r-1}\, b(n,1)$ *for* $r \geq 2$ *and* $n \geq 3$.
(2) $c(n+1,2) \leq c(n,1) + 1$ *for* $n \geq 3$.
(3) $c(n+1,r) \leq c(n,1)$ *for* $r \geq 3$ *and* $n \geq 3$.
(4) $\tilde{b}(n+1) \leq 2\, b(n,1)$ *for* $n \geq 3$.

Proof. Let Z be a Fano $(n+1)$-fold of the first species with index $r > 1$. By the assumption, the linear system $|(r-1)H|$ contains a smooth divisor X. By adjunction theorem and Lefschetz hyperplane theorem,

X is a Fano n-fold of the first species with index one. Note

$$(-K_X)^n = (r-1)H^{n+1} = (r-1)\left(-\frac{1}{r}K_Z\right)^{n+1}.$$

Therefore we have

$$(-K_Z)^{n+1} = \frac{r^{n+1}}{r-1}(-K_X)^n \leq \frac{r^{n+1}}{r-1}\, b(n,1),$$

which leads to the first inequality. Taking the coholomology of the sheaf sequence

$$0 \to \mathcal{O}_Z((2-r)H) \to \mathcal{O}_Z(H) \to \mathcal{O}_X(H|_X) \to 0,$$

we obtain an exact sequence

$$0 \to H^0(\mathcal{O}_Z((2-r)H)) \to H^0(\mathcal{O}_Z(H)) \to H^0(\mathcal{O}_X(H|_X)) \to H^1(\mathcal{O}_Z((2-r)H)).$$

By the Lefschectz hyperplane theorem, $H|_X$ is the fundamental divisor of X. If $r = 2$, then $h^0(\mathcal{O}_Z(H)) = h^0(\mathcal{O}_X(H|_X)) + 1$, which leads to the second inequality. If $r > 2$, then $H^0(\mathcal{O}_Z((2-r)H)) = 0$ and $H^1(\mathcal{O}_Z((2-r)H)) = 0$, where the latter holds due to the Serre duality and Kodaira vanishing theorem. Therefore $h^0(\mathcal{O}_Z(H)) = h^0(\mathcal{O}_X(H|_X))$. So we have the third inequality.

Let $\mathcal{F}$ be a family Fano $(n+1)$-fold that is of cyclic type with index one. Then it contains a Fano $(n+1)$-fold W that is an m-cover over a Fano $(n+1)$-fold V. Let $\phi : W \to V$ be the covering map and S be the branch locus. Then S is linearly equivalent to $-mkH$ for some positive integer k, where H is the fundamental divisor of V. Note $K_W = \phi^*((m-1)kH + K_V) = ((m-1)k - r_V)\phi^*(H)$, where r_V be the index of V. Note that $r_V \geq 2$. Since W has index one, we have $((m-1)k - r_V) = -1$. Note

$$m - 1 \leq (m-1)k = r_V - 1.$$

So $m \leq r_V$, i.e. $\frac{m}{r_V} \leq 1$. Note that $\frac{r_V}{r_V - 1} \leq 2$.

Hence

$$\begin{aligned}
(-K_W)^{n+1} &= (-(m-1)k + r_V)^{n+1}(\phi^* H)^{n+1} \\
&= mH^{n+1} \\
&= \frac{m(-K_V)^{n+1}}{r_V^{n+1}} \\
&\leq \frac{(-K_V)^{n+1}}{r_V^{n}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{b(n+1, r_V)}{r_V^n} \\
&\leq \frac{r_V}{r_V - 1} b(n,1) \qquad (\because \text{ inequality } (1)) \\
&\leq 2\, b(n,1),
\end{aligned}
$$

which leads to the fourth inequality. □

It is known that $b(4) = 5^4$ ([10]) and $b(5) \leq 9^5$([11]). So under the assumption, we have some bounds for Fano 5-folds and 6-folds:

(1) $b(5,r) \leq 5^4 \cdot \frac{r^6}{r-1}$ for $r \geq 2$,
(2) $\tilde{b}(5) \leq 2 \cdot 5^4$,
(3) $b(6,r) \leq 9^5 \cdot \frac{r^7}{r-1}$ for $r \geq 2$,
(4) $\tilde{b}(6) \leq 2 \cdot 9^5$.

By the Lefschetz hyperplane theorem and the adjunction formula, more data from Fano n-fold of the first species with index one can be transferred to the Fano $(n+1)$-folds of the first species with index greater than one, such as some of Hodge numbers, cupproduct structure of Chern classes and integral cohomology classes.

References

[1] V. A. Iskovskih, *Fano threefolds. I*, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), no. 3, 516-562, 717.

[2] V. V. Shokurov, *Smoothness of a general anticanonical divisor on a Fano variety,* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 2, 430-441.

[3] V. V. Shokurov, *The existence of a line on Fano varieties,* Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 4, 922-964, 968.

[4] P. M. H. Wilson, *Fano fourfolds of index greater than one,* J. Reine Angew. Math. **379** (1987), 172-181.

[5] S. Mukai, *Biregular classification of Fano 3 -folds and Fano manifolds of coindex 3,* Proc. Nat. Acad. Sci. U.S.A. **86** (1989), no. 9, 3000-3002.

[6] F. Campana, *Un theoreme de finitude pour les varietes de Fano suffisamment unireglees,* Geometrie complexe (Paris, 1992), 23–33, Actualites Sci. Indust., 1438, Hermann, Paris, 1996.

[7] S. Kobayashi and T. Ochiai, *Characterizations of complex projective spaces and hyperquadrics,* J. Math. Kyoto Univ. **13** (1973), 31-47.

[8] T. Fujita, *On the structure of polarized manifolds with total deficiency one. I, II, III.* J. Math. Soc. Japan **32** (1980), no. 4, 709-725.

[9] M. Mella, *Existence of good divisors on Mukai varieties,* J. Algebraic Geom. **8** (1999), no. 2, 197-206.

[10] J.-M. Hwang, *On the degrees of Fano four-folds of Picard number 1,* J. Reine Angew. Math. **556** (2003), 225-235.

[11] K. Watanabe, *Lengths of chains of minimal rational curves on Fano manifolds,* J. Algebra **325** (2011), 163-176.

*
Department of Mathematics Education
Hongik University
Seoul 121-791, Republic of Korea
E-mail: nhlee@hongik.ac.kr